Individual claims reserving using the Aalen-Johansen estimator

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This presentation is about the work in Bladt and Pittarello (2023).

Martin Bladt, Associate Professor. University of Copenhagen.

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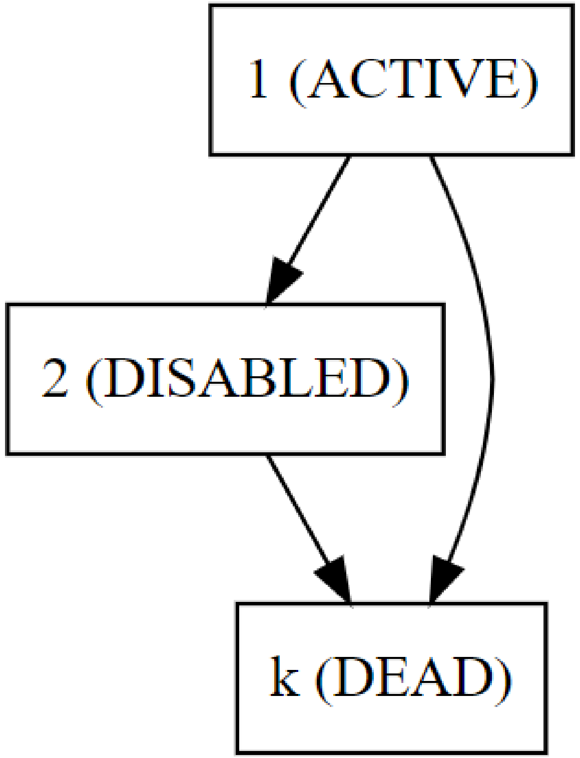
Literature Review

- MSM have found widespread use in the domain of life insurance (Hoem 1969).



Example of MSM for modeling biometric states (active, deceased).

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- Notable exceptions in non-life insurance (Hesselager 1994; Maciak, Mizera, and Pešta 2022).

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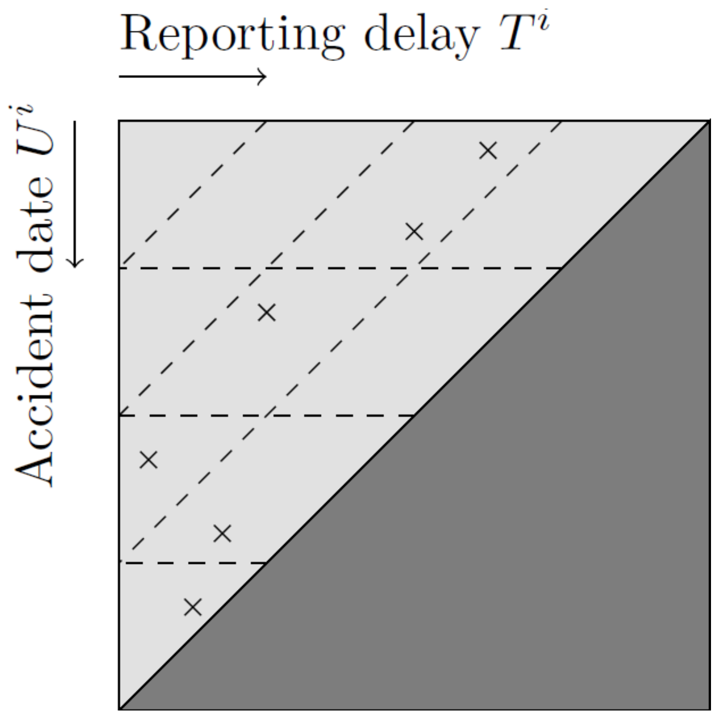


The development triangle of reporting delays

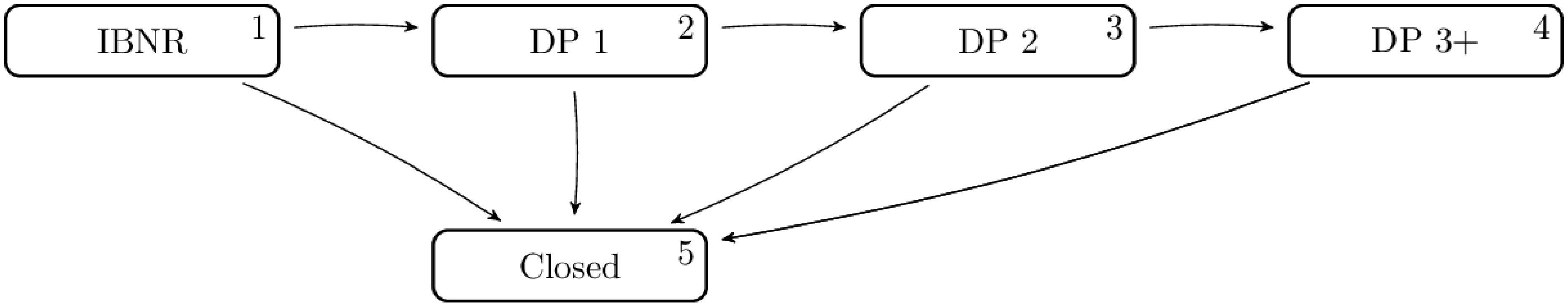
$$\mathcal{D}^{(k-1)} = \{d_{\ell,j} : \ell + j \leq k; \ell, j = 1, \dots, k-1\},$$

stems from the i.i.d. data $(U^i, T^i), i = 1, \dots, n$:

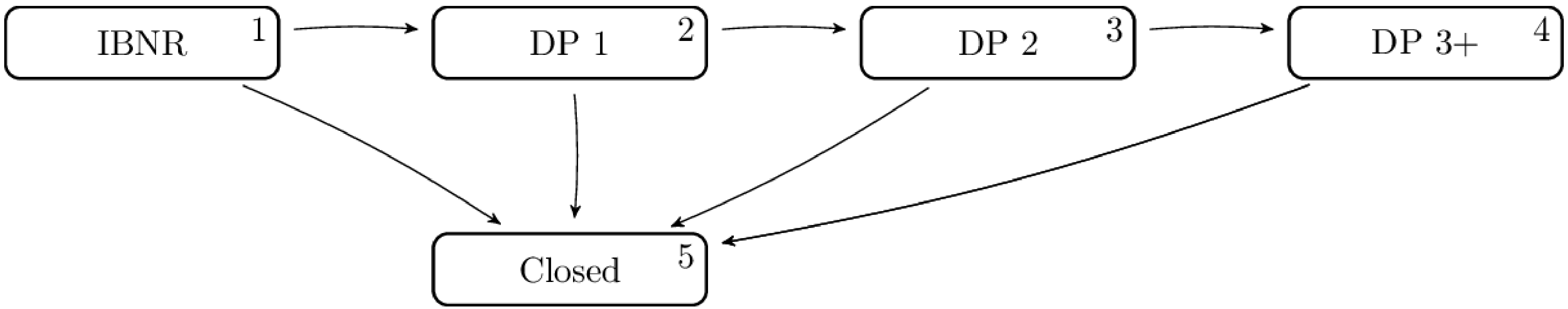
- U^i accident date.
- T^i the delay between accident and report.
- $d_{\ell,j} = \sum_{i=1}^{n} 1_{\{U^i=\ell \text{ and } T^i=j\}}$ denoting the total claims reported in accident period ℓ with delay j.

A MSM for individual claims reserving

We model based on a continuous-time non-explosive pure jump process denoted by J on a finite state space, $\mathcal{S} = \{1, \ldots, k\}, k \in \mathbb{N}$. States correspond to the development periods (DP's) within a development triangle, and the "time spent" between state transitions corresponds to the claim size growth between DP's.



Example of multi-state model for claims reserving, with $k = 5$.



Example of multi-state model for claims reserving, with $k = 5$.

We introduce a strictly positive random variable W describing right-censoring. We are interested the triplet

$$\left(X, (J_z)_{0 \leqslant z \leqslant W}, Y \wedge W\right).$$

We denote by Y the possibly infinite absorption time of J, and J_z the state occupied by j in z.

For any x in the support of X we model the conditional occupation probabilities

$$p_k(z \mid x) = E\left[I(\{J_z = k\}) \mid X = x\right].$$

Our sample is constituted of the i.i.d. replicates

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Define

- $\delta^i := I(Y^i \leq W^i)$, which equals 1 when the observation is absorbed (closed claim).
- $n = n^{\texttt{Closed}} + n^{\texttt{RBNS}}$

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Under certain regulatory conditions (Bladt and Pittarello 2023), we derive the conditional Aalen-Johansen estimator

$$p^{(n)}(z \mid x) = p^{(n)}(0 \mid x)\Pi^z \left(\mathrm{Id} + \Lambda^{(n)}(\mathrm{d}s \mid x)\right),$$

where $\mathrm{p}_j^{(n)}(0 \mid x) = I_j^{(n)}(0 \mid x)$, and $\Lambda^{(n)}(\mathrm{d}s \mid x)$ is the conditional Nelson-Aalen estimator for cumulative hazard Bladt and Furrer (2023). We denote the product integral with Π.

Predictors

We are presently interested in predicting the ultimate cost of our claims,

$$Y^{\texttt{Closed}} + Y^{\texttt{RBNS}} := \sum_{i=1}^{n} \delta^i Y^i + \sum_{i=1}^{n} (1 - \delta^i) Y^i,$$

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The predictor of the final total cost of RBNS claims is defined by

$$\hat{Y}^{\texttt{RBNS}} = \sum_{i}^{n} I(\delta^i = 0) \left(W^i + \hat{E}[Y|Y > W^i, X = x] \right)$$

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The general formula for predicting the m-th moment is explicitly given by

$$\hat{E}\left[Y^m | Y > W^i, X = x\right] = \frac{1}{1 - \mathrm{p}_k^{(n)}(W^i \mid x)} \int_{W^i}^{+\infty} m y^{m-1} (1 - \mathrm{p}_k^{(n)}(y \mid x)) \mathrm{d}y.$$

A model for IBNR claims

We describe the total cost of IBNR claims with the collective risk model in (Klugman, Panjer, and Willmot 2012, 715:Ch.9):

$$Y^{\texttt{IBNR}} = \sum_{i'=1}^{n^{\texttt{IBNR}}} \tilde{Y}^{i'}.$$

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The expected cost of IBNR claims is estimated by

$$\hat{Y}^{\texttt{IBNR}} = \hat{E}[Y^{\texttt{IBNR}}] = \hat{n}^{\texttt{IBNR}} \hat{E}[\tilde{Y}],$$

with the m-th moment of $\tilde{Y}$ estimated, using the **unconditional** Aalen-Johansen, by

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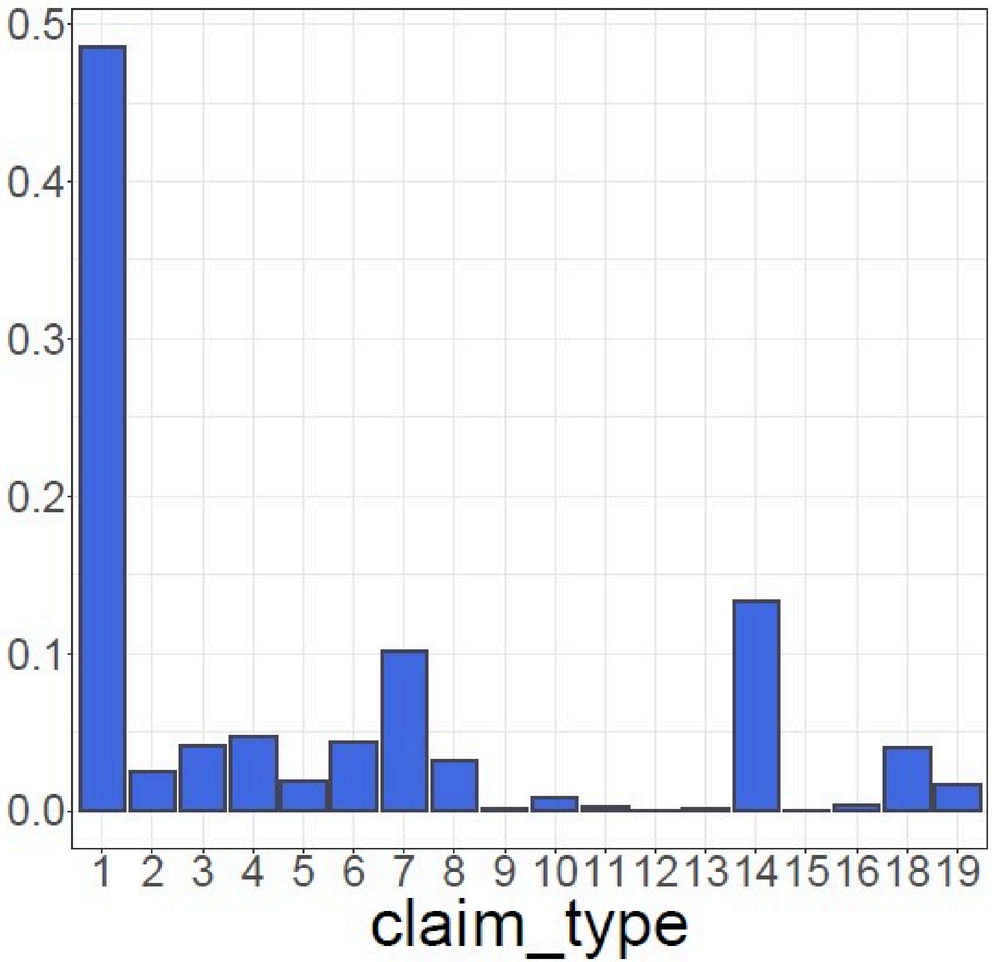
We calibrate the Mack Chain-Ladder estimator on $\mathcal{D}^{(k-1)}$ for $\hat{n}^{\texttt{IBNR}}$.

The total size of claims is $Y^{\texttt{TOT}} = Y^{\texttt{Closed}} + Y^{\texttt{RBNS}} + Y^{\texttt{IBNR}}$ and we estimate it with $\hat{Y}^{\texttt{TOT}} = Y^{\texttt{Closed}} + \hat{Y}^{\texttt{RBNS}} + \hat{Y}^{\texttt{IBNR}}$.

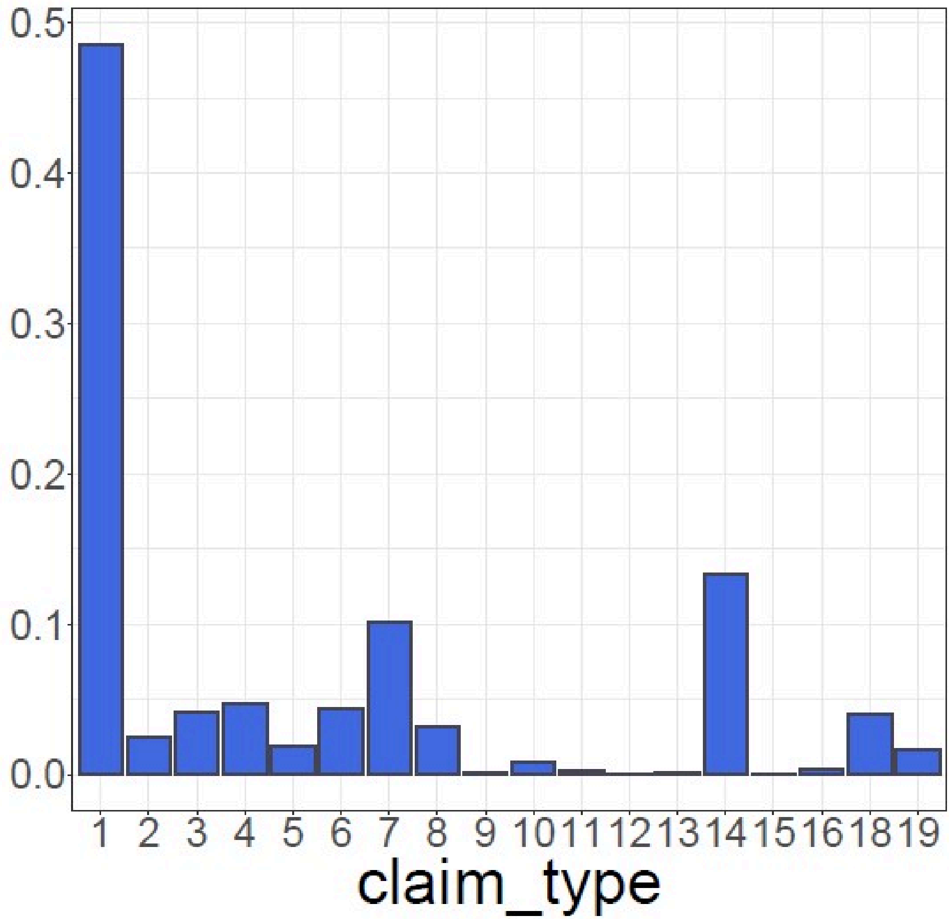
A data application on an insurance portfolio

A real data set from a Danish insurance company.

Covariates	**Description**
Claim_number	Policy identifier
claim_type $\in \{1, \ldots, 20\}$	Type of claim
AM	Accident month
CM	Calendar month of report
DM	Development month
incPaid	Incremental paid amount
Delta	Indicator, 0 when the claim is open



Frequency by `claim_type`.



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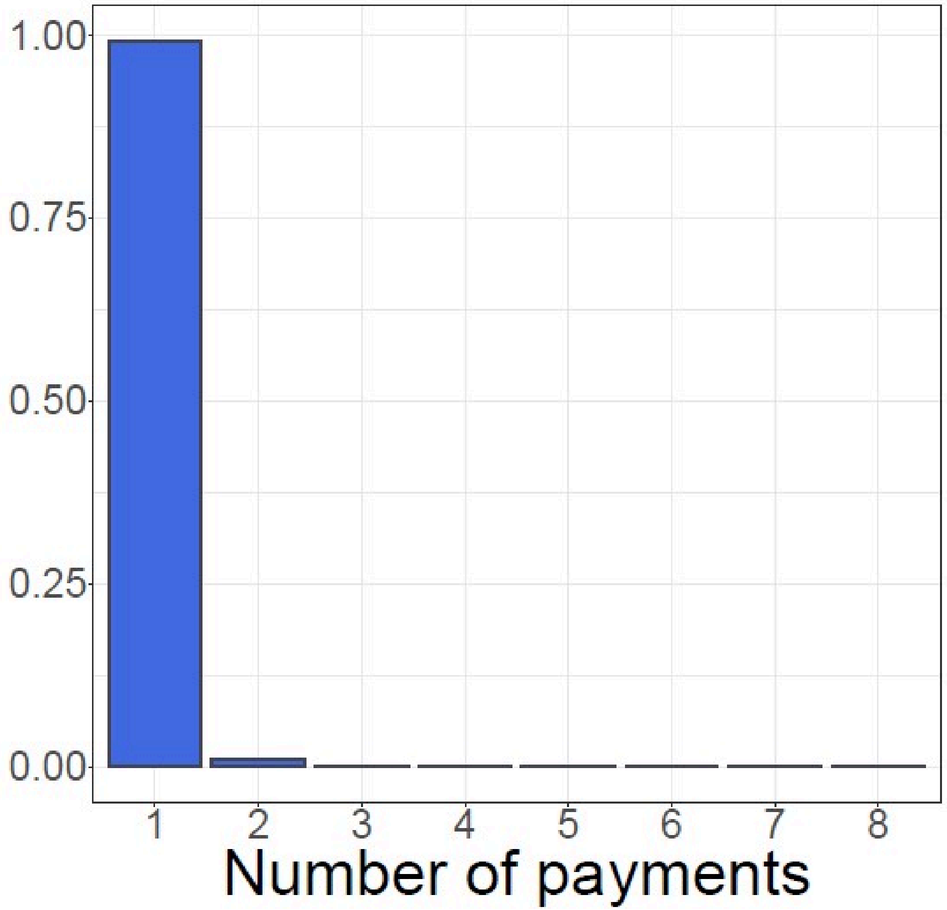
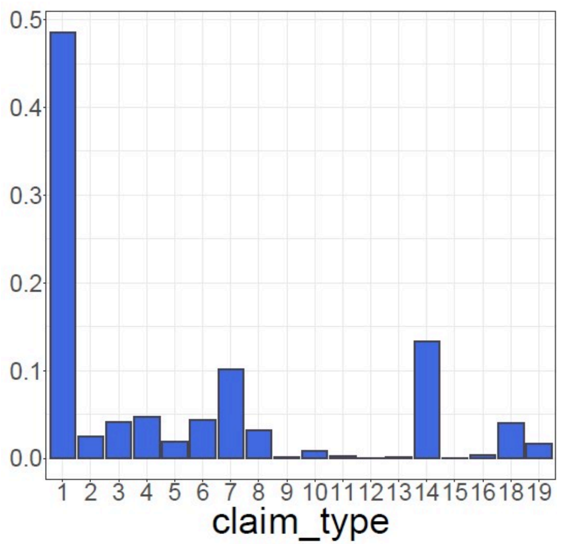


Distribution of the number of payments.



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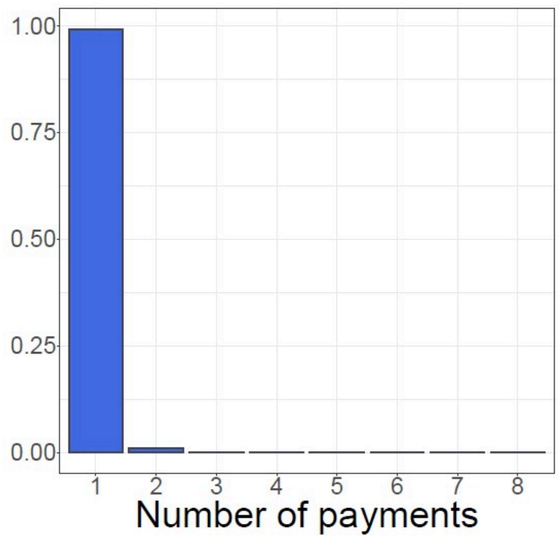

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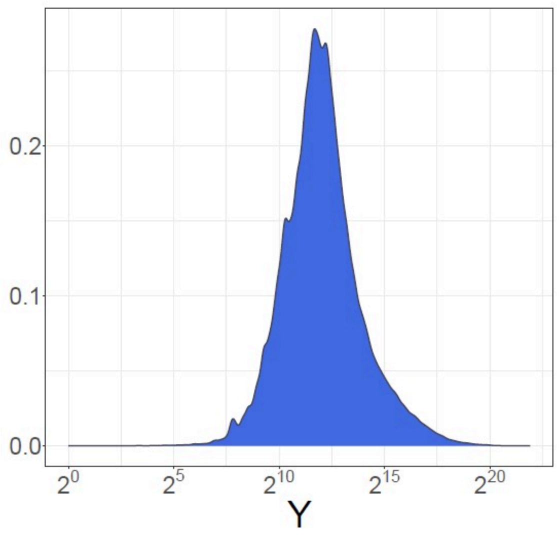

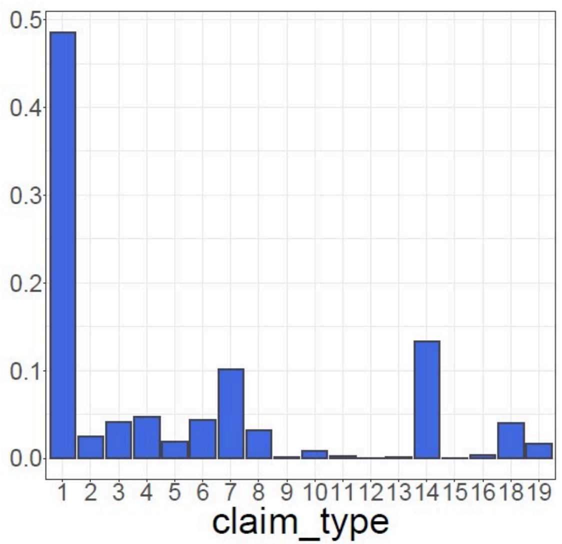

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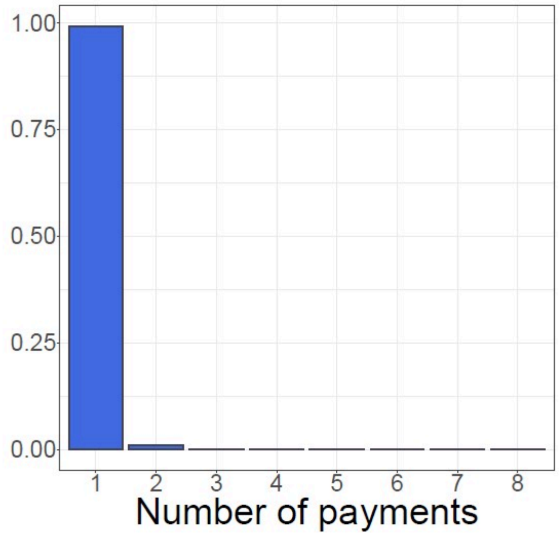

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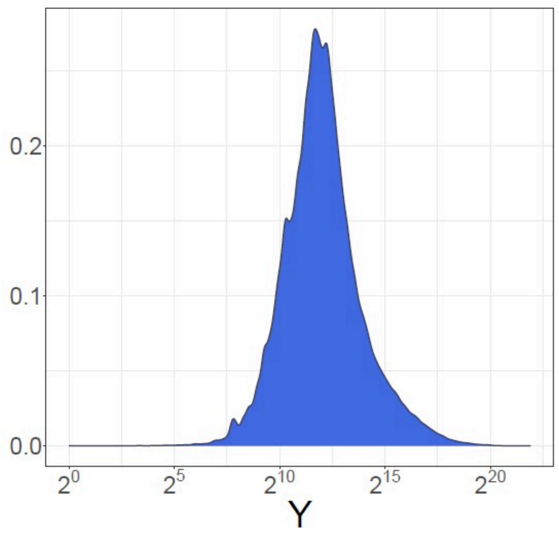

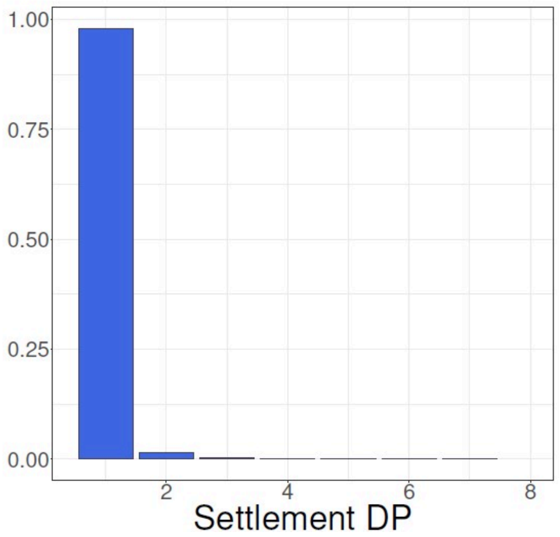

Model comparison on different datasets

Model strategy:

1. We cut our time-serie at different depths ($k = 4, 5, 6, 7$).

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1. We cut our time-serie at different depths ($k = 4, 5, 6, 7$).
2. We define two performance metrics (next slide).
3. For the different choices of k compare:
    - Our model (AJ) conditioning on the feature `claim_type`.
    - Our model (AJ) **without** conditioning on the feature `claim_type`.
    - The Chain-Ladder model (CL, Mack 1993).

- The (average) Continuously Ranked Probability Score (CRPS, Gneiting and Raftery (2007)).

$$\mathrm{CRPS}(\mathrm{p}_k^{(n)}(z \mid x), y) = \int_0^{+\infty} (\mathrm{p}_k^{(n)}(z \mid x) - 1_{\{y \leq z\}})^2 \mathrm{d}z$$

- The (average) Continuously Ranked Probability Score (CRPS, Gneiting and Raftery (2007)).
- The error incidence,

$$\mathtt{EI} = \frac{\hat{Y}^{\mathtt{TOT}}}{Y^{\mathtt{TOT}}} - 1.$$

Models comparison for $k = 4, 5, 6, 7$

k	`claim_type`	$Y^{\texttt{TOT}}$	$\mathrm{EI}^{\mathrm{TOT}}$ (AJ)	$\mathrm{EI}^{\mathrm{TOT}}$ (CL)	$\sqrt{\hat{\mathrm{Var}}(Y^{\texttt{TOT}})}/\hat{Y}^{\texttt{TOT}}$ (AJ)	$\sqrt{\hat{\mathrm{Var}}(Y^{\texttt{TOT}})}/\hat{Y}^{\texttt{TOT}}$ (CL)	CRPS
4	✓	616.1327	0.0035	0.0157	0.0029	0.0023	1.0000
	✗		-0.0029		0.0029		1.1403
5	✓	822.5956	0.0064	0.0209	0.0008	0.0024	1.0000
	✗		-0.0061		0.0007		0.4596
6	✓	999.6005	0.0059	0.0173	0.0017	0.0017	1.0000
	✗		-0.0052		0.0017		0.9987
7	✓	1190.9112	0.0146	0.0144	0.0011	0.0017	1.0000
	✗		-0.0142		0.0011		1.0022

Models comparison for $k = 4, 5, 6, 7$

k	`claim_type`	Y^{TOT}	EI^{TOT} (AJ)	EI^{TOT} (CL)	$\sqrt{\hat{\text{Var}}(Y^{\text{TOT}})}/\hat{Y}^{\text{TOT}}$ (AJ)	$\sqrt{\hat{\text{Var}}(Y^{\text{TOT}})}/\hat{Y}^{\text{TOT}}$ (CL)	CRPS
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	✗		-0.0052		0.0017		0.9987
7	✓	1190.9112	0.0146	0.0144	0.0011	0.0017	1.0000
	✗		-0.0142		0.0011		1.0022

Notes on the computation of $\text{Var}\widehat{(Y^{\text{TOT}})}$

We compute numerically the AJ estimator for the standard deviation,

$$\text{Var}\widehat{(Y^{\text{TOT}})} = \text{Var}\widehat{(Y^{\text{RBNS}})} + \text{Var}\widehat{(Y^{\text{IBNR}})},$$

as sum of the individual variability.

In the CL case, we use the Mack estimator for the process variance.

An individual model for the claim size

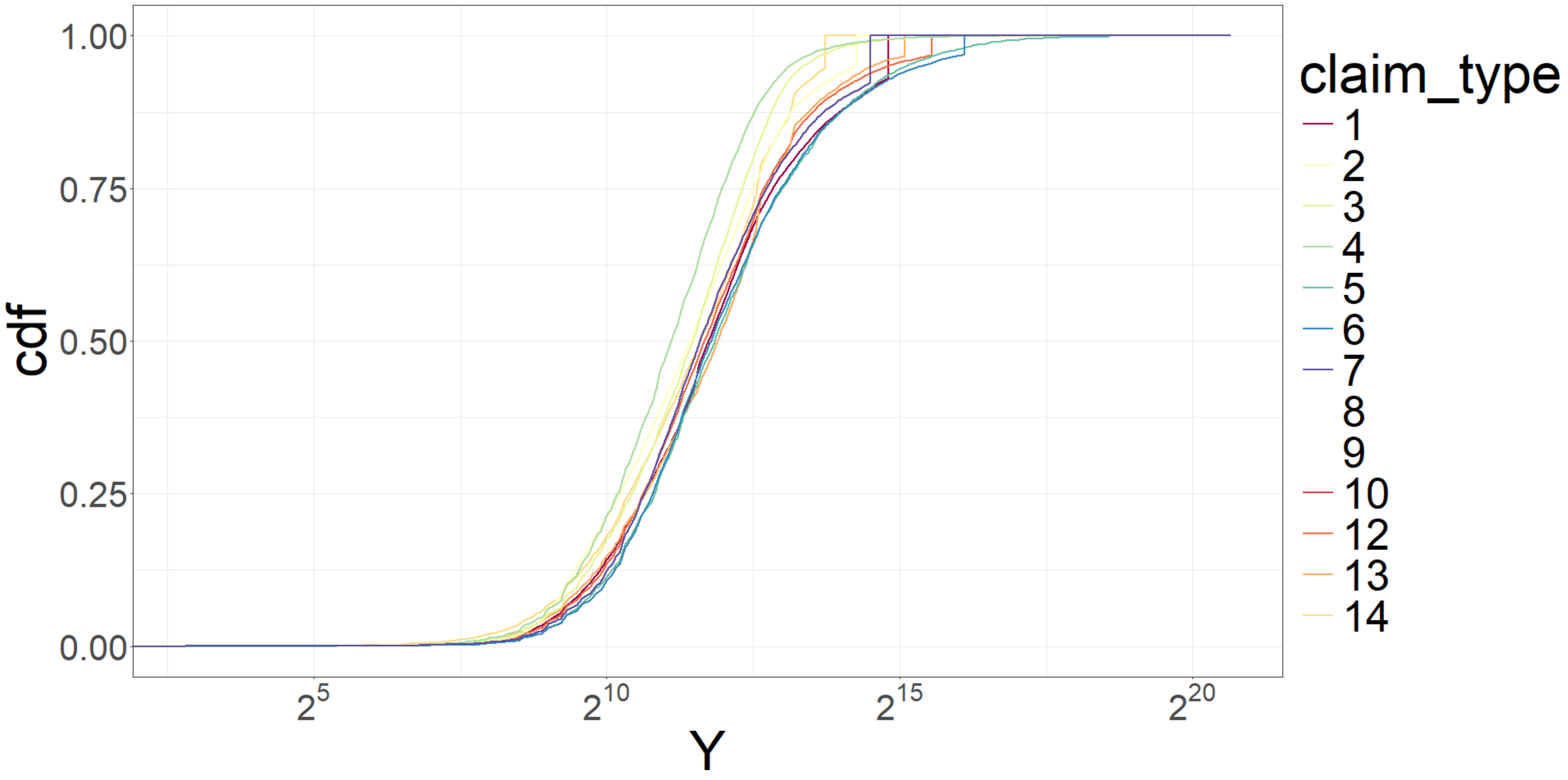

Figure 1: Severity curve for a k=4.

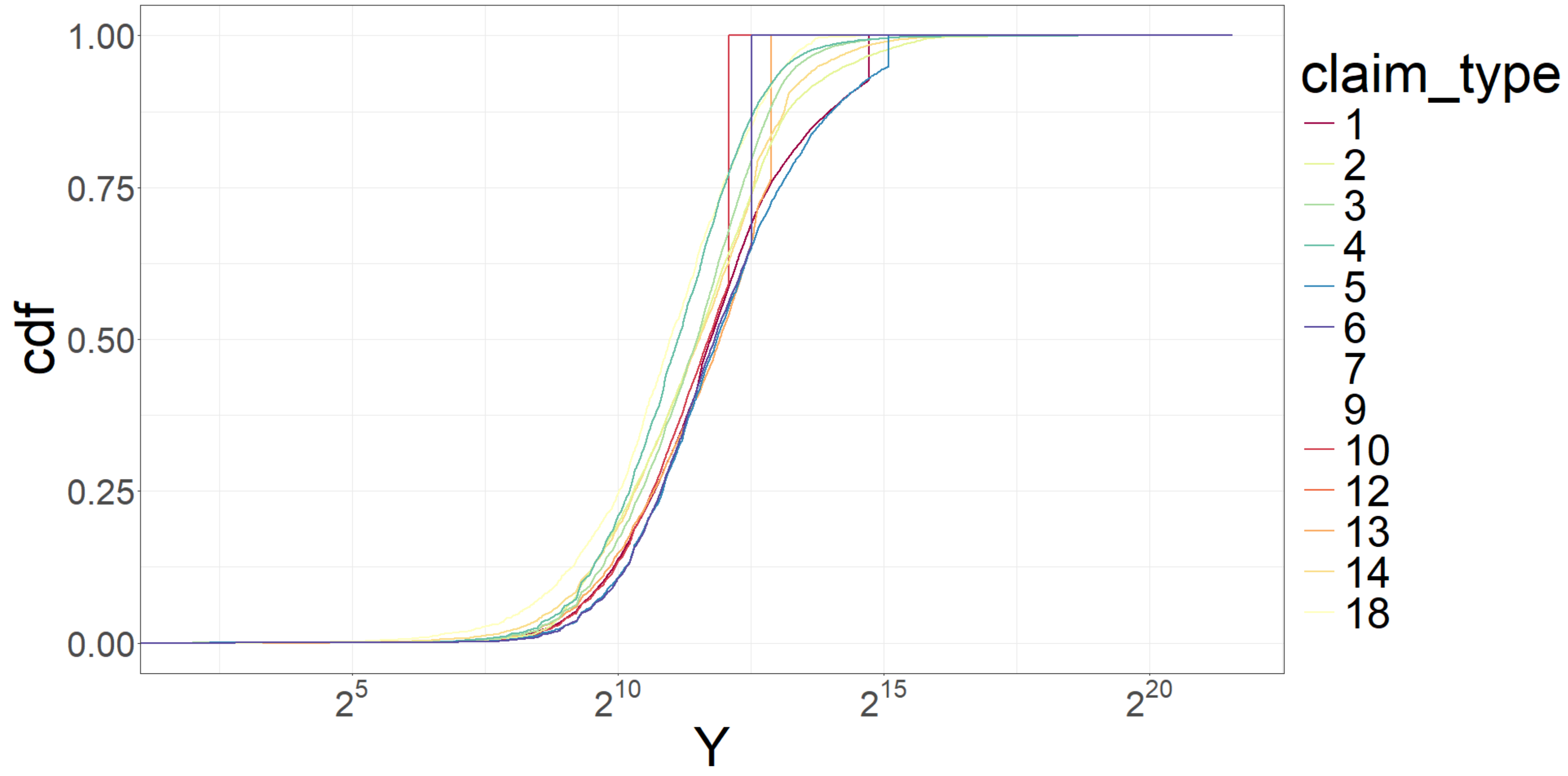

Figure 2: Severity curve for a k=5.

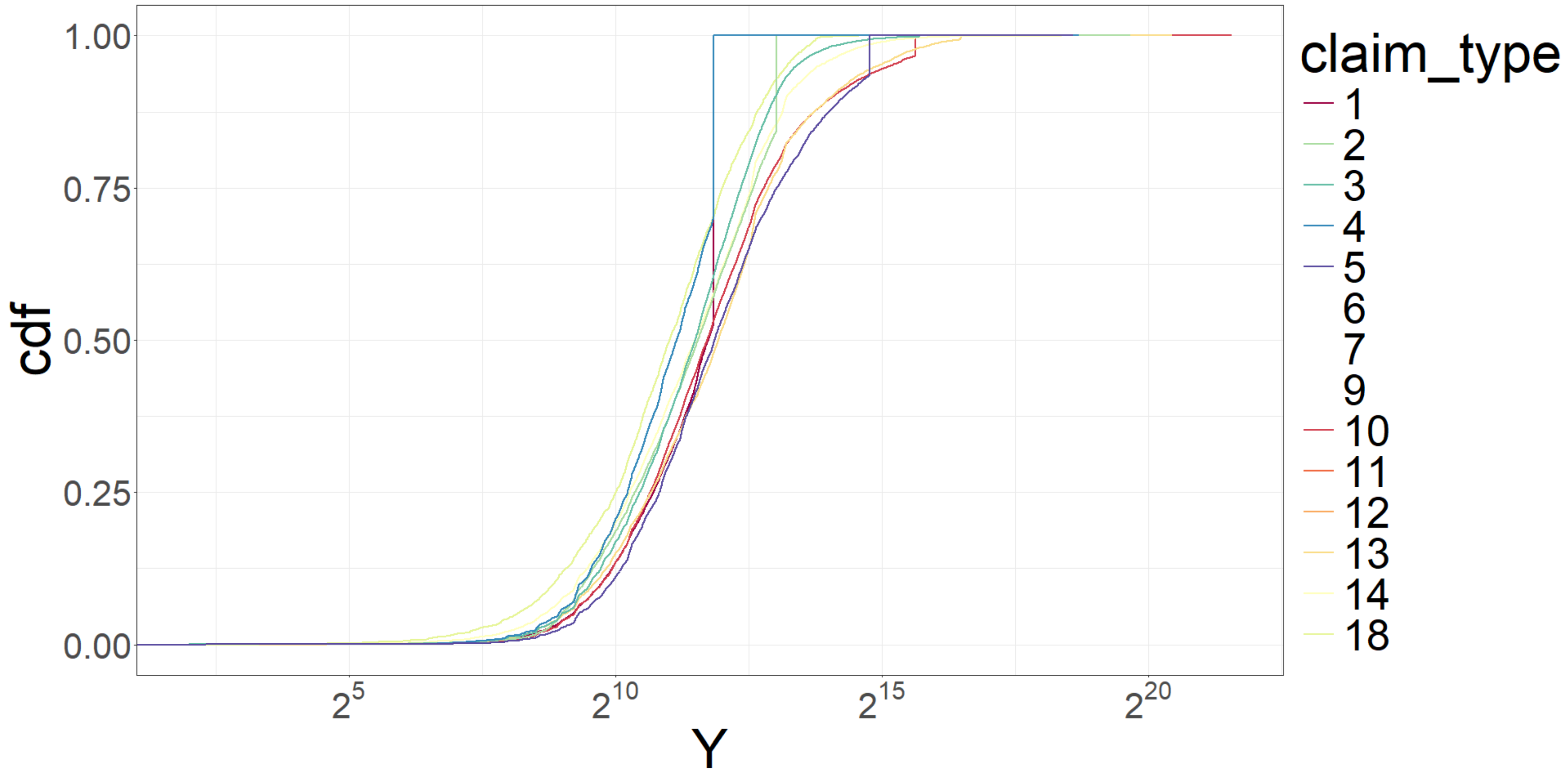

Figure 3: Severity curve for a k=6.

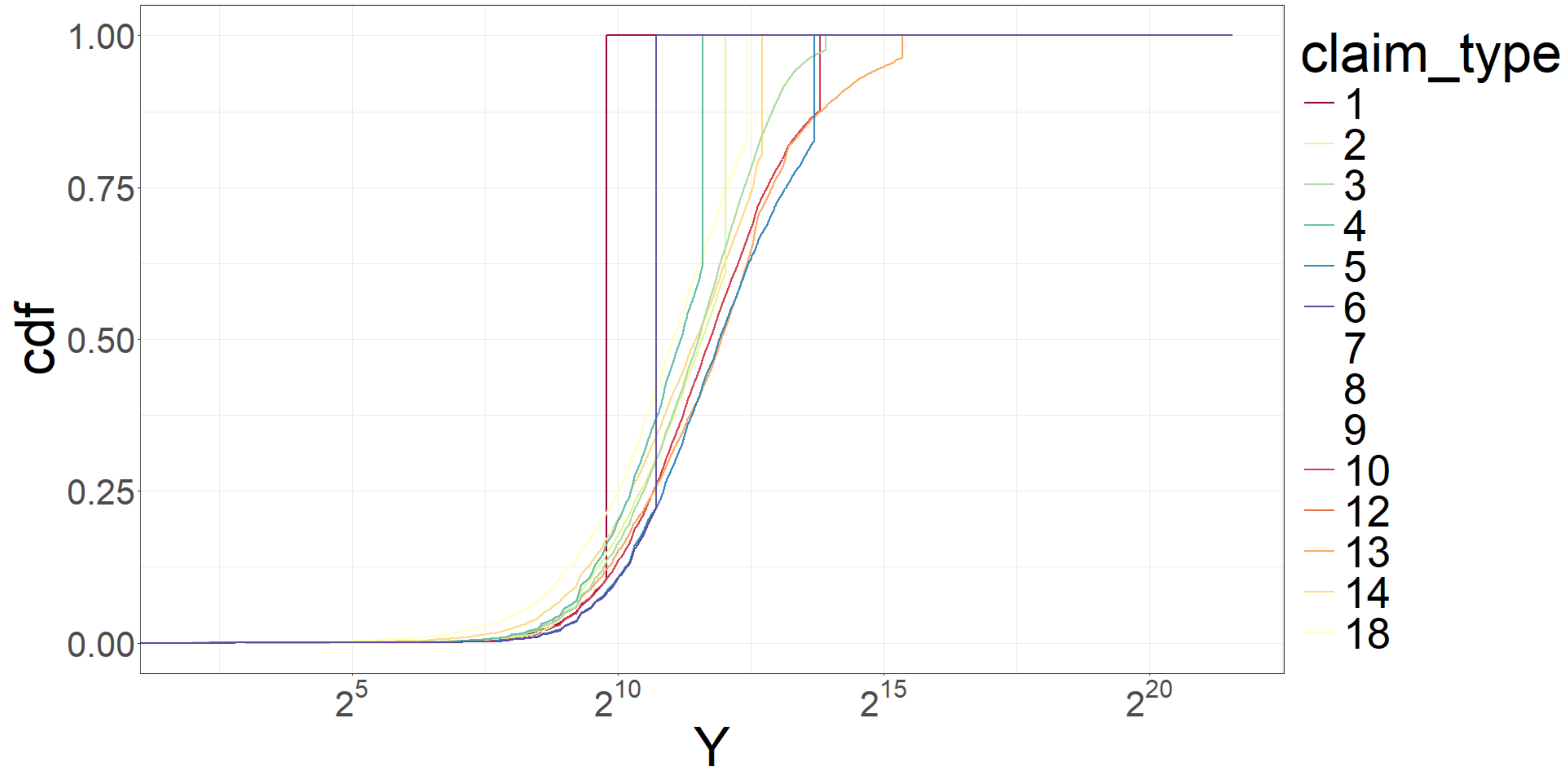

Figure 4: Severity curve for a k=7.

Model comparison on a single dataset

Model strategy:

1. We cut our time-serie at maximum of 5 calendar periods.

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1. We cut our time-serie at maximum of 5 calendar periods.
2. We compare, using CRPS and `EI`:
    - The AJ model conditioning on the feature `claim_type` for $k = 4, 5, 6$.
    - The AJ model **without** conditioning on the feature `claim_type` for $k = 4, 5, 6$.
    - The Chain-Ladder model (CL, Mack 1993).

Results for the data set with $k = 6$.

k	`claim_type`	`EI` (AJ)	$\sqrt{\hat{\text{Var}}(Y^{\texttt{TOT}})}/\hat{Y}^{\texttt{TOT}}$ (CL) (AJ)	CRPS (average, relative)
4	✗	-0.0059	0.0016	1.0143
	✓	-0.0040	0.0020	1.0000
5	✗	-0.0069	0.0015	0.9916
	✓	-0.0052	0.0018	1.0000
6	✗	-0.0052	0.0017	0.9987
	✓	-0.0059	0.0017	1.0000

- $Y^{\texttt{TOT}} = 999.6005$ (Actual, millions).
- $\sqrt{\hat{\text{Var}}(Y^{\texttt{TOT}})}/\hat{Y}^{\texttt{TOT}} = 0.0017$ (CL).
- $\texttt{EI} = 0.0173$ (CL).
- Using the CRPS we can select the model with features for $k = 4$:
  - Claims reserve 1.485 millions.
  - Standard Deviation 2.0385 millions.

Replicable results

- Replication material can be found in our GitHub folder.

README

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Individual claims reserving using the Aalen-Johansen estimator

This repository contains the code to replicate the manuscript Individual claims reserving using the Aalen-Johansen estimator.

The replication material is organized as described in this file and it can be used for replicating exactly the results of our manuscript.

The script `elaborate_latex_tables.R` contains the code to print the tables in the manuscript from the the results in the `results_csv` folder.

The script `helper_functions_ajr.R` contains functions that we used to perform the calculations.

The real data that we used in our application will not be shared, but we provide the results that we obtained.

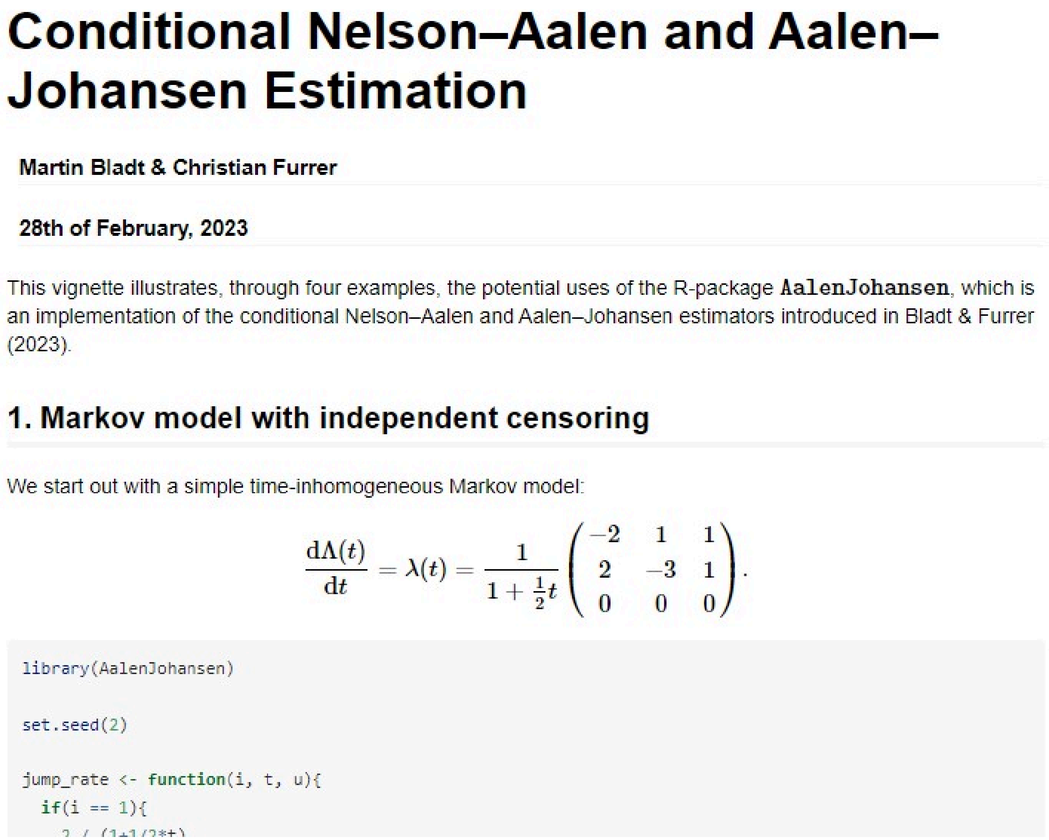
For each section, we describe below the relevant replication material.

Section 4. Simulation of RBNS claims

Section 4.1. Implementation

gpitt71/conditional-aj-reserving folder ReadME.md.

- Replication material can be found in our GitHub folder.
- The R package `AalenJohansen` for the conditional Aalen-Johansen estimation is available on CRAN.



Conditional Nelson–Aalen and Aalen–Johansen Estimation

Martin Bladt & Christian Furrer

28th of February, 2023

This vignette illustrates, through four examples, the potential uses of the R-package **AalenJohansen**, which is an implementation of the conditional Nelson–Aalen and Aalen–Johansen estimators introduced in Bladt & Furrer (2023).

1. Markov model with independent censoring

We start out with a simple time-inhomogeneous Markov model:

$$\frac{\mathrm{d}\Lambda(t)}{\mathrm{d}t} = \lambda(t) = \frac{1}{1+\frac{1}{2}t}\begin{pmatrix} -2 & 1 & 1 \\ 2 & -3 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

```
library(AalenJohansen)

set.seed(2)

jump_rate <- function(i, t, u){
  if(i == 1){
    2 / (1+1/2*t)
```

`AalenJohansen` package vignette.

Thank you for your attention!

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