Auto-Calibration and Isotonic Recalibration

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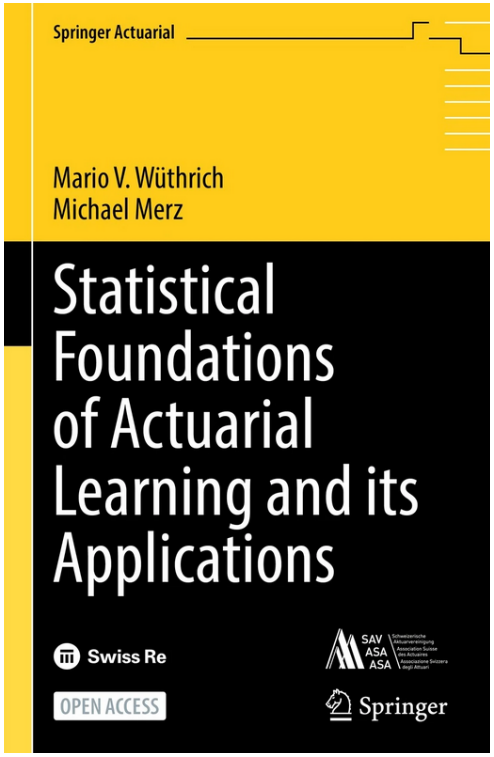
Overview

- Introduction: regression models
- Auto-calibration and isotonic recalibration
- Conclusions



- **Introduction: regression models**

Best-estimate premium

- For known data generating model, compute true best-estimate of a policyholder with features $\boldsymbol{x}$ by

$$\mu^*(\boldsymbol{x}) := \mathbb{E}\left[Y \middle| \boldsymbol{x}\right].$$

- For unknown data generating model, estimate μ^* from a sample $(Y_i, \boldsymbol{x}_i)_{i=1}^n$ that has been generated by this unknown model. Solve

$$\widehat{\mu} \;=\; \underset{\mu \in \mathcal{M}}{\arg\min}\, \frac{1}{n} \sum_{i=1}^n L\left(Y_i, \mu(\boldsymbol{x}_i)\right),$$

  for a given model class $\mathcal{M}$, and for a strictly consistent loss function L for mean estimation; see Gneiting (2011).

- Popular model selection criteria:
  - ⋆ out-of-sample losses (under strictly consistent loss functions);
  - ⋆ auto-calibration and global balance property;
  - ⋆ conditional T-reliability diagrams and score decompositions.

- **Auto-calibration and isotonic recalibration**

Auto-calibration

- Literature on auto-calibration: Schervish (1989), Menon et al. (2012), Tsyplakov (2013), Gneiting–Ranjan (2013), Pohle (2020), Gneiting–Resin (2022); Tasche (2021), Krüger–Ziegel (2021); Denuit et al. (2021), Fissler et al. (2022), W. (2023), Lindholm et al. (2023), W.–Ziegel (2023), ...

Regression function $\boldsymbol{x} \mapsto \mu(\boldsymbol{x})$ is auto-calibrated for $(Y, \boldsymbol{x})$ if, a.s.,

$$\mu(\boldsymbol{x}) = \mathbb{E}\left[Y \mid \mu(\boldsymbol{x})\right].$$

▷ This means that every price cohort $\mu(\boldsymbol{x})$ is on average self-financing,

or, in other words, there is no systematic cross-financing between different price cohorts $\mu(\boldsymbol{x}) \neq \mu(\boldsymbol{x}')$ within the portfolio.

▷ Insurance price systems should generally fulfill this important property!

Empirical testing for auto-calibration

- **Binning** (Hosmer–Lemeshow (1980) χ^2-test, Lindholm et al. (2023)): Build disjoint price intervals (bins) $I_k = [a_k, a_{k+1})$ and consider the average claim in each bin

$$\frac{\sum_{i=1}^n Y_i \, \mathbb{1}_{\{\mu(\boldsymbol{x}_i)\in I_k\}}}{\sum_{i=1}^n \mathbb{1}_{\{\mu(\boldsymbol{x}_i)\in I_k\}}} \overset{??}{\approx} \frac{\sum_{i=1}^n \mu(\boldsymbol{x}_i) \, \mathbb{1}_{\{\mu(\boldsymbol{x}_i)\in I_k\}}}{\sum_{i=1}^n \mathbb{1}_{\{\mu(\boldsymbol{x}_i)\in I_k\}}}.$$

- **Local Regression** (Loader (1999), Denuit et al. (2021)): Binning is a discretized version of a local regression (or kernel smoother) that regresses the responses Y from the price cohorts $\mu(\boldsymbol{x})$

$$\texttt{locfit}(Y \sim \mu(\boldsymbol{x}),\ \texttt{alpha} = 0.1,\ \texttt{deg} = 2).$$

- Both methods are sensitive to hyperparameter selection.

Isotonic recalibration

- Isotonic regression is a non-parametric way to restore the auto-calibration property.

Isotonic regression solves the optimization problem (for positive weights w_i)

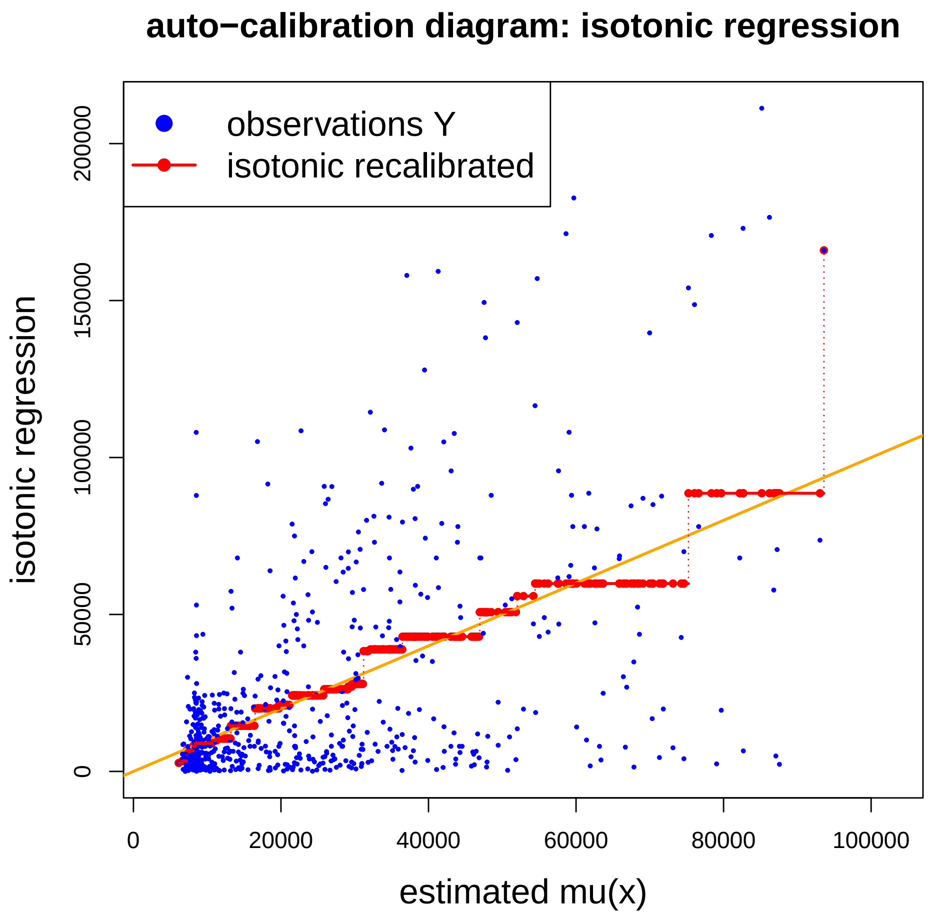
$$\widehat{\boldsymbol{m}} = \underset{\boldsymbol{m}=(m_1,\ldots,m_n)^\top \in \mathbb{R}^n}{\arg\min} \sum_{i=1}^{n} w_i \left(Y_i - m_i\right)^2,$$

subject to $m_k \leq m_j \iff \mu(\boldsymbol{x}_k) \leq \mu(\boldsymbol{x}_j)$.

- Isotonic regression preserves the ordering in $\mu(\boldsymbol{x}_i)_{i=1}^n$. This requires that the first regression function $\mu(\cdot)$ provides (approximately) the correct ordering.

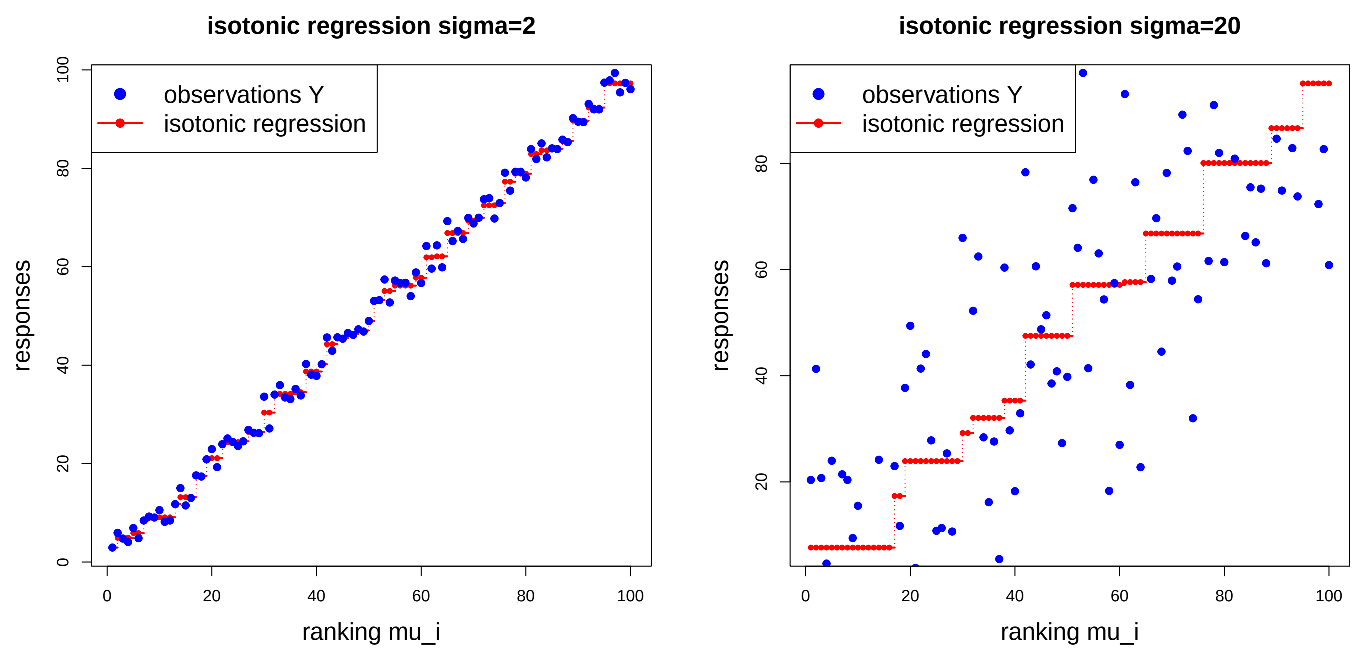
- Isotonic regression $(Y_i, \widehat{m}_i)_{i=1}^n = (Y_i, \widehat{m}(\boldsymbol{x}_i))_{i=1}^n$ is (empirically) auto-calibrated.

Isotonic recalibration: step function



- Isotonic regression: natural (optimal) binning without hyperparameter tuning.

Isotonic regression and signal-to-noise ratio



(left): high signal-to-noise ratio (right): low signal-to-noise ratio

- **Theorem** (W.–Ziegel, 2023)**.** The expected number of steps in the isotonic (step) regression function is increasing in the signal-to-noise ratio.

- Low signal-to-noise ratio: isotonic regression leads to low complexity partition of the feature space $\Rightarrow$ explainability.

- **Conclusions**

Concluding remarks

- Estimation should be based on strictly consistent loss functions; Gneiting (2011).
- Bregman divergences are the only strictly consistent loss functions for mean estimation, Savage (1971) and Gneiting (2011).
- Any regression function should be auto-calibrated for insurance pricing.
- Isotonic recalibration restores the auto-calibration property (empirically).
- A low signal-to-noise ratio leads to a low complexity isotonic recalibrated functions.
- A low complexity partition of the feature space gives explainability.

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