Deep Hedging of Long-Term Financial Derivatives

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Topic

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Deep hedging of long-term financial derivatives☆

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- Topic: Optimal hedging of **long-term** European financial derivatives with deep reinforcement learning.
- Implementation: github.com/alexandrecarbonneau.

Motivation

This paper studies the problem of global hedging **very long-term European derivatives** (many years) with dynamic hedging.

- Such long maturity derivatives are analogous, under some assumptions, **to financial guarantees** sold with equity-linked insurance products.
- This study examines exclusively the mitigation of **financial risk exposure**.

Global risk minimization

For $\delta := \{\delta_n\}_{n=0}^{N}$ assets positions at each time-step, S_N the underlying price of a European derivative of payoff $\Phi(S_N, Z_N)$ and portfolio value V_N^{δ}:

$$\delta^{\star} = \arg\min_{\delta} \mathbb{E}\left[\mathcal{L}(\Phi(S_N, Z_N) - V_N^{\delta})\right], \quad (1)$$

where $\mathcal{L} : \mathbb{R} \to \mathbb{R}$ is the penalty or loss function.

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- Downside: **Penalizes equally gains and losses**.

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- Penalizes **only hedging losses**, not gains.

Contributions of the paper

1) Perform extensive Monte Carlo experiments for the risk mitigation of long-term lookback options with global hedging.

2) Provide qualitative insights into specific characteristics of the optimized long-term global policies (i.e. neural networks).

Deep hedging

• For several popular (Markov) dynamics of the traded assets, the optimal policy has the form

$$\delta^{\star}_{n+1} = f(n, V_n, S_n, Z_n, \mathcal{I}_n)$$

for some function f and $\mathcal{I}_n$ containing additional relevant information.

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- Approximate f with a neural network F_θ with parameters θ:

$$f(n, V_n, S_n, \mathcal{I}_n, Z_n) \approx F_\theta(n, V_n, S_n, \mathcal{I}_n, Z_n), \tag{2}$$

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- Optimization problem **boils down to optimizing** θ:

$$\min_{\delta} \mathbb{E}\left[\mathcal{L}\left(\Phi(S_N, Z_N) - V_N^{\delta}\right)\right] \approx \min_{\theta} \mathbb{E}\left[\mathcal{L}\left(\Phi(S_N, Z_N) - V_N^{\delta^\theta}\right)\right] \quad (3)$$

where δ^θ is to be understood as the output of F_θ.

Market setup

- Market setup considered is essentially the same as in Coleman et al. (2007).
- **Lookback option** to hedge with a time-to-maturity of 10 years with payoff $\Phi(S_N, Z_N) = \max(Z_N - S_N, 0)$.
- **Vanilla calls and puts** used for hedging have a time-to-maturity of 1 year, are traded once and held until expiration.
- **Merton Jump-Diffusion** for underlying.

Global quadratic vs semi-quadratic - MJD

Table 6: Benchmarking of quadratic deep hedging and semi-quadratic deep hedging to hedge the lookback option of $T = 10$ years under the MJD model.

Statistics	Mean	RMSE	semi-RMSE	$\text{VaR}_{0.95}$	$\text{VaR}_{0.99}$	$\text{CVaR}_{0.95}$	$\text{CVaR}_{0.99}$
$\mathcal{L}^{MSE}$							
Stock (year)	−1.6	19.8	15.6	32.3	66.4	54.5	95.4
Stock (month)	0.2	11.2	9.4	15.7	42.8	32.6	64.6
Two options	0.0	5.2	3.8	6.7	15.4	12.7	25.1
Six options	−0.1	1.3	0.9	1.4	3.6	2.9	6.2
$\mathcal{L}^{SMSE}$							
Stock (year)	−35.2	49.7	6.7	11.4	31.7	24.6	47.7
Stock (month)	−22.8	33.8	4.2	6.5	18.3	14.3	29.6
Two options	−5.9	11.2	1.7	2.2	7.1	5.5	12.2
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- Downside risk reduction improvement with $\mathcal{L}^{\text{SMSE}}$ over $\mathcal{L}^{\text{MSE}}$ ranges between 45% to 76%.

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- Downside risk reduction improvement with $\mathcal{L}^{\text{SMSE}}$ over $\mathcal{L}^{\text{MSE}}$ ranges between 45% to 76%.
- Hedging gains across all hedging instruments with $\mathcal{L}^{\text{SMSE}}$.

Benchmarking takeaway

With semi-quadratic global hedging:

- **Tailor-made** to match the financial objectives of hedgers.
  - Smallest downside risk metrics + significant hedging gains across all benchmarks.
- Conclusion: **should be prioritized** (when possible) over other dynamic hedging procedures considered in this study.

Conclusion

- Present a reinforcement learning approach relying on the class of deep hedging algorithms to hedge long-term European financial derivatives.

- Perform extensive benchmarking of global policies for long-term lookback put options over many different scenarios (e.g. hedging instruments and jump risk).

- Numerical results clearly demonstrate the vast superiority of non-quadratic global hedging.

  $\rightarrow$ **downside risk** two to three times smaller than best benchmark.

  $\rightarrow$ **hedging gains** on average.

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